Nonholonomic Ricci Flows, Exact Solutions in Gravity, and Symmetric and Nonsymmetric Metrics

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Abstract We provide a proof that nonholonomically constrained Ricci flows of (pseudo) Riemannian metrics positively result into nonsymmetric metrics (as explicit examples, we consider flows of some physically valuable exact solutions in general relativity). There are constructed and analyzed three classes of solutions of Ricci flow evolution equations defining nonholonomic deformations of Taub NUT, Schwarzschild, solitonic and pp-wave symmetric metrics into nonsymmetric ones.

Keywords Nonsymmetric metrics · Nonholonomic manifolds · Nonlinear connections · Nonholonomic Ricci flows · Taub NUT spacetimes · Solitons in gravity · pp-Waves

1 Introduction

One of the most remarkable results in modern mathematics following from the theory of the Ricci flows [1, 2] is the proof of the Poincaré conjecture by Grisha Perelman [3–5]. It states that every closed smooth simply connected three-dimensional manifold is topologically equivalent to a sphere. In a more general context, the Perelman's results complete the Hamilton's program on Ricci flows, settle the second major conjecture (by Thurston) in geometry and topology, and show a number of ways for further progress and applications in mathematics and physics, see detailed reviews of results in Refs. [6–11].

It was shown in Refs. [10, 11] that Ricci flows of the (pseudo) Riemannian metrics may result not only in generalized Lagrange and/or Finsler like geometries, and inversely, but also in nonholonomic configurations enabled with nonsymmetric metrics if the evolution equations are subjected to certain type of nonholonomic (nonintegrable) constraints. Here, for our purposes, we cite the review [12] as the basic reference on modelling such geometries

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on nonholonomic manifolds and monographs [13, 14] on Lagrange–Finsler spaces defined in metric compatible form on tangent bundles.

The surprising results that nonholonomic Ricci flows naturally relate the class of (pseudo) Riemannian metrics to various types of geometries described by locally anisotropic and/or nonsymmetric metrics and generalized connection structures follow from the evolution equations of geometric objects and nonholonomic distributions. In such cases, one considers flows on nonholonomic manifolds enabled with nonholonomic distributions inducing locally fibred structures into conventional horizontal (h) and vertical (v) directions. One could be developed a corresponding Ricci flow theory of metrics and nonholonomic distributions on Riemann–Cartan (in particular, (pseudo) Riemannian) spaces possessing nontrivial torsion structure defined in a metric compatible form by a linear connection. The corresponding nontrivial nonholonomic distributions are defined by generic off-diagonal metrics establishing a "preferred" nonholonomic frame structure with associated nonlinear connection (N-connection).[1]

On nonholonomic manifolds, one can be constructed two classes of "remarkable" metric compatible linear connections: the Levi Civita and the so-called canonical distinguished connection (d-connection) which are completely defined by a chosen metric structure. The first linear connection is torsionless and the second one, vanishing on the globalized h- and v-distributions, contains some nonzero h-v-coefficients induced by the N-connection/off-diagonal metric coefficients.

In general, the Ricci tensor for the canonical d-connection is nonsymmetric even it is defined by a symmetric metric structure. Together with the evolution of N-connection structure, this results in nontrivial nonsymmetric components of metrics induced by Ricci flows of geometric objects on usual (pseudo) Riemannian spaces [10, 11]. The constructions hold true for any models of symmetric and/or nonsymmetric theory of gravity which provides an additional geometrical strong argument for geometrical and physical models with non-symmetric metrics and nonholonomic structures.[2] Such results can be obtained only from evolution equations for fundamental geometric objects (like metrics and linear and nonlinear connections) not involving gravitational and matter field equations.

It should be noted that none physical principle prohibits us to consider theories with nonsymmetric metrics [15, 16] and the geometry of such spaces has a long and interesting history of development and applications in physics and mechanics. There are known the A. Einstein's attempts to generalize his theory in order to unify gravity with electromagnetism (when the nonsymmetric part of metric had been identified with the electromagnetic field strength tensor), see Ref. [17], and then to elaborate a unified theory of physical fields by introducing a complex metric field with Hermitian symmetry, see [18].

Then, L.P. Eisenhart has investigated the geometric properties of the so-called generalized Riemannian spaces with nonsymmetric metrics when the symmetric part is nondegenerated [19, 20]. He dealt with the problem of the linear connections which are compatible

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[1]The generic off-diagonal metrics can not be diagonalized by coordinate transforms; we can work equivalently with any system of local frames or coordinates but geometrically it is preferred to elaborate the constructions in a form adapted to the N-connection structure.

[2]It should be noted that the main results in the (holonomic, if to follow a terminology oriented to nonholonomic generalizations) Hamilton–Perelman Ricci flow theory were derived under the assumptions that the (pseudo) Riemannian/Kählerian metrics will flow positively into other (pseudo) Riemannian/Kählerian metrics and the evolution equations are not subjected to additional constraints. Further generalizations are possible for flows of geometric objects and structures when various classes of nonholonomic restrictions are introduced into consideration and the spacetime geometry is not constrained to be only of symmetric/commutative... (pseudo) Riemannian type.

with a general (nonsymmetric) metric structure (it is called the Eisenhart problem). It was solved in an important particular case in [21] and retaken for the generalized Lagrange and Finsler spaces in [22]; a review of such results is contained in Chapter 8 of monograph [13]. The nonsymmetric gravity theory and its generalizations [15, 16, 23–25] with applications in modern astrophysics and cosmology [26, 27] consist a well defined and perspective direction in gravity and field interactions theory and mathematical physics.

The aim of this paper is to provide a geometric motivation for gravity models with nonsymmetric metrics following Ricci flow theory. We also show how the anholonomic frame method of constructing solutions evolution equations, see Refs. [28–32], can be applied for generating solitonic pp-wave nonsymmetric deformations of Taub NUT and Schwarzschild metrics. The second partner of this article [33] is devoted to the geometry of nonholonomic manifolds enabled with nonsymmetric metric and nonlinear connection structure.

The work is organized as follows: In Sect. 2, we consider the evolution equations in Ricci flow theory with nonholonomic constrains resulting in nonsymmetric metrics. Section 3 is devoted to the anholonomic frame method in constructing exact solutions in gravity with symmetric and nonsymmetric metric components and generalization of the approach for generating solutions for evolution equations with nonholonomic constraints. In Sect. 4, we construct a general class of solutions describing how nonholonomic deformations of four dimensional Taub NUT spaces result in nonsymmetric metrics; we derive Ricci flow scenaria when the evolution parameter is identified with the time like coordinate and the geometry is defined by off-diagonal metrics and pp-wave configurations. We analyze nonholonomic and nonsymmetric Ricci flows of Schwarzschild metrics induced by solitonic pp-waves, when the evolution parameter is not related to spacetime coordinates, in Sect. 5. Finally, we present conclusions and comment on some further perspectives in Sect. 6.

2 Nonsymmetric Ricci Flows

The aim of this section is to provide a geometric formulation for the systems of evolution equations with nonholonomic Ricci flows transforming symmetric metrics into nonsymmetric ones. We develop the results stated by Theorem 4.3 in Ref. [10]) and formulas (22)–(24) in Ref. [11]). The reader may see additional discussions and details on the geometry of nonlinear connections and nonholonomic manifolds and applications to physics in [12] and introduction sections in Ref. [10, 11]. In monograph [34], there are contained the bulk of proofs for geometric formulas and differential and tensor calculus adapted to the nonlinear connection structure.

2.1 Preliminaries: N-Anholonomic Manifolds

Let $\mathbf{V}$ be a four dimensional nonholonomic manifold enabled with nonlinear connection (N-connection) structure

$$\mathbf{N} = N_i^a(u)dx^i \otimes \frac{\partial}{\partial y^a} \tag{1}$$

defining a holonomic-nonholonomic splitting of dimension when $n + m = 2 + 2$, when the tangent bundle $T\mathbf{V}$ splits as a Whitney sum

$$T\mathbf{V} = h\mathbf{V} \oplus v\mathbf{V} \tag{2}$$

into corresponding "horizontal" and "vertical" subspaces $h\mathbf{V}$ and $v\mathbf{V}$. The local coordinates on $\mathbf{V}$ are denoted in the form $u = (x, y)$, or $u^\alpha = (x^i, y^a)$, where the "horizontal" indices run the values $i, j, k, \ldots = 1, 2, \ldots, n$ and the "vertical" indices run the values $a, b, c, \ldots = n + 1, n + 2, \ldots, n + m$.[3] The N-connection (1) states on $\mathbf{V}$ a preferred frame structure $\mathbf{e}_\nu = (\mathbf{e}_i, e_a)$, where

$$\mathbf{e}_i = \frac{\partial}{\partial x^i} - N_i^a(u)\frac{\partial}{\partial y^a} \quad \text{and} \quad e_a = \frac{\partial}{\partial y^a}, \tag{3}$$

and the dual frame (coframe) structure $\mathbf{e}^\mu = (e^i, \mathbf{e}^a)$, where

$$e^i = dx^i \quad \text{and} \quad \mathbf{e}^a = dy^a + N_i^a(u)dx^i. \tag{4}$$

These formulas can be written in matrix forms,

$$\mathbf{e}_\alpha = \mathbf{e}_\alpha^{\underline{\alpha}}(u)\,\partial_{\underline{\alpha}} \quad \text{and} \quad \mathbf{e}^\beta = \mathbf{e}_{\underline{\beta}}^\beta(u)\,du^{\underline{\beta}},$$

where

$$\mathbf{e}_\alpha^{\underline{\alpha}} = \begin{bmatrix} \delta_i^{\underline{i}} & N_i^b(u)\,\delta_b^{\underline{a}} \\ 0 & \delta_a^{\underline{a}} \end{bmatrix}, \qquad \mathbf{e}_{\underline{\beta}}^\beta = \begin{bmatrix} \delta_{\underline{i}}^i(u) & -N_k^b(u)\,\delta_{\underline{i}}^k \\ 0 & \delta_{\underline{a}}^a \end{bmatrix}, \tag{5}$$

when $\delta_i^{\underline{i}}$ is the Kronecker delta function.[4] The vielbeins (4) satisfy the nonholonomy relations

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha\mathbf{e}_\beta - \mathbf{e}_\beta\mathbf{e}_\alpha = W_{\alpha\beta}^\gamma\mathbf{e}_\gamma \tag{6}$$

with (antisymmetric) nontrivial anholonomy coefficients $W_{ia}^b = \partial_a N_i^b$ and $W_{ji}^a = \Omega_{ij}^a$, where

$$\Omega_{ij}^a = \frac{\partial N_i^a}{\partial x^j} - \frac{\partial N_j^a}{\partial x^i} + N_i^b\frac{\partial N_j^a}{\partial y^b} - N_j^b\frac{\partial N_i^a}{\partial y^b} \tag{7}$$

is the curvature of N-connection.

A distinguished symmetric metric (in brief, symmetric d-metric) on a N-anholonomic manifold $\mathbf{V}$ is a usual second rank symmetric tensor $\mathbf{g}$ which with respect to a N-adapted basis (4) can be written in the form

$$\mathbf{g} = g_{ij}(x, y)\,e^i \otimes e^j + h_{ab}(x, y)\,\mathbf{e}^a \otimes \mathbf{e}^b. \tag{8}$$

With respect to a local coordinate basis $du^\alpha = (dx^i, dy^a)$, this metric can be equivalently written in the form

$$\mathbf{g} = \underline{g}_{\alpha\beta}(u)\,du^\alpha \otimes du^\beta,$$

where

$$\underline{g}_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b h_{ab} & N_j^e h_{ae} \\ N_i^e h_{be} & h_{ab} \end{bmatrix}. \tag{9}$$

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[3] For the tangent bundle, $\mathbf{V} = TM$, we can consider that both type of indices run the same values.

[4] We shall use always "boldface" symbols if it would be necessary to emphasize that certain spaces (geometrical objects) are provided (adapted) with (to) a N-connection structure. With respect to N-adapted bases, we can introduce respectively, distinguished vectors, tensors, spinors, ..., in brief, d-vectors, d-tensors, d-spinors, ...

In a more general case, one can be considered nonsymmetric metric structures $\check{\mathbf{g}} = \mathbf{g} + \mathbf{a}$, when (for instance, in local form)

$$\underline{\check{g}}_{\alpha\beta} = \underline{g}_{\alpha\beta} + \underline{a}_{\alpha\beta}, \tag{10}$$

$$\underline{g}_{\alpha\beta} = \underline{g}_{\beta\alpha}, \quad \underline{a}_{\alpha\beta} = -\underline{a}_{\beta\alpha}.$$

The decomposition into symmetric and anti-symmetric components holds true with respect to any local bases including the N-adapted ones.

A distinguished connection (d-connection) $\mathbf{D}$ on a N-anholonomic manifold $\mathbf{V}$ is a linear connection conserving under parallelism the Whitney sum (2). One writes that $\mathbf{D} = (hD, vD)$, or $\mathbf{D}_\alpha = (D_i, D_a)$, when the coefficients with respect to N-adapted basis (3) and (4) are parametrized in the form $\mathbf{D} = \{\boldsymbol{\Gamma}^\gamma_{\alpha\beta} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc})\}$, with $hD = (L^i_{jk}, L^a_{bk})$ and $vD = (C^i_{jc}, C^a_{bc})$.

The torsion of a d-connection $\mathbf{D} = (hD, vD)$ for any d-vectors $\mathbf{X}, \mathbf{Y}$ is defined by d-tensor field

$$\mathbf{T}(\mathbf{X}, \mathbf{Y}) \doteq \mathbf{D_X Y} - \mathbf{D_Y X} - [\mathbf{X}, \mathbf{Y}]. \tag{11}$$

The nontrivial torsion coefficients are parametrized in the form

$$\mathbf{T} = \{\mathbf{T}^\alpha_{\beta\gamma} = -\mathbf{T}^\alpha_{\gamma\beta} = (T^i_{jk}, T^i_{ja}, T^a_{jk}, T^b_{ja}, T^b_{ca})\},$$

where

$$T^i_{jk} = L^i_{jk} - L^i_{kj}, \quad T^i_{ja} = -T^i_{aj} = C^i_{ja}, \quad T^a_{ji} = \Omega^a_{ji},$$

$$T^a_{bi} = -T^a_{ib} = \frac{\partial N^a_i}{\partial y^b} - L^a_{bi}, \quad T^a_{bc} = C^a_{bc} - C^a_{cb}, \tag{12}$$

can be computed by a d-form calculus for $\boldsymbol{\Gamma}^\alpha_\beta = \boldsymbol{\Gamma}^\alpha_{\beta\gamma}\mathbf{e}^\gamma$, with the coefficients defined with respect to (4) and (3), when $\mathbf{T} = \{\mathcal{T}^\alpha\}$,

$$\mathcal{T}^\alpha \doteq \mathbf{De}^\alpha = d\mathbf{e}^\alpha + \boldsymbol{\Gamma}^\alpha_\beta \wedge \mathbf{e}^\beta. \tag{13}$$

By a straightforward d-form calculus, we can find the N-adapted components of the curvature $\mathbf{R} = \{\mathcal{R}^\alpha_\beta\}$, when

$$\mathbf{R}(\mathbf{X}, \mathbf{Y}) \doteq \mathbf{D_X D_Y} - \mathbf{D_Y D_X} - \mathbf{D}_{[\mathbf{X}, \mathbf{Y}]}, \tag{14}$$

with

$$\mathcal{R}^\alpha_\beta \doteq \mathbf{D}\boldsymbol{\Gamma}^\alpha_\beta = d\boldsymbol{\Gamma}^\alpha_\beta - \boldsymbol{\Gamma}^\gamma_\beta \wedge \boldsymbol{\Gamma}^\alpha_\gamma = \mathbf{R}^\alpha_{\beta\gamma\delta}\mathbf{e}^\gamma \wedge \mathbf{e}^\delta, \tag{15}$$

when $\mathbf{R}^\alpha_{\beta\gamma\delta}$ splits into N-adapted components:

$$R^i_{hjk} = e_k L^i_{hj} - e_j L^i_{hk} + L^m_{hj} L^i_{mk} - L^m_{hk} L^i_{mj} - C^i_{ha}\Omega^a_{kj},$$

$$R^a_{bjk} = e_k L^a_{bj} - e_j L^a_{bk} + L^c_{bj} L^a_{ck} - L^c_{bk} L^a_{cj} - C^a_{bc}\Omega^c_{kj},$$

$$R^i_{jka} = e_a L^i_{jk} - D_k C^i_{ja} + C^i_{jb} T^b_{ka}, \tag{16}$$

$$R^c_{bka} = e_a L^c_{bk} - D_k C^c_{ba} + C^c_{bd} T^c_{ka},$$

$$R^i_{jbc} = e_c C^i_{jb} - e_b C^i_{jc} + C^h_{jb} C^i_{hc} - C^h_{jc} C^i_{hb},$$

$$R^a_{\ bcd} = e_d C^a_{\ bc} - e_c C^a_{\ bd} + C^e_{\ bc} C^a_{\ ed} - C^e_{\ bd} C^a_{\ ec}.$$

Contracting respectively the components of (16), one proves that the Ricci tensor $\mathbf{R}_{\alpha\beta} \doteq \mathbf{R}^\tau_{\ \alpha\beta\tau}$ is characterized by h- v-components, i.e. d-tensors,

$$R_{ij} \doteq R^k_{\ ijk}, \quad R_{ia} \doteq -R^k_{\ ika}, \quad R_{ai} \doteq R^b_{\ aib}, \quad R_{ab} \doteq R^c_{\ abc}. \qquad (17)$$

It should be noted that this tensor is not symmetric for arbitrary d-connections $\mathbf{D}$.

From the class of arbitrary d-connections $\mathbf{D}$ on $\mathbf{V}$, one distinguishes those which are metric compatible (metrical d-connections) satisfying the condition $\mathbf{Dg} = \mathbf{0}$ including all h- and v-projections

$$D_j g_{kl} = 0, \quad D_a g_{kl} = 0, \quad D_j h_{ab} = 0, \quad D_a h_{bc} = 0.$$

We emphasize that in this work we define the metric compatibility with respect to the symmetric part of a metric, i.e. with respect to $\mathbf{g}$, considering that the antisymmetric part $\mathbf{a}$ will be induced noholonomically by Ricci flows, also by $\mathbf{g}$. In a more general case, it is possible from the very beginning to work with $\breve{\mathbf{g}}$, see discussion in Ref. [33].

The Levi Civita linear connection $\triangledown = \{_\lfloor\Gamma^\alpha_{\ \beta\gamma}\}$ is uniquely defined by the symmetric metric structure (9) by the conditions $_\lfloor\mathcal{T} = 0$ and $\triangledown\mathbf{g} = 0$. It should be noted that this connection is not adapted to the distribution (2) because it does not preserve under parallelism the h- and v-distribution.

One exists a N-adapted equivalent of the Levi Civita connection $\triangledown$, called the canonical d-connection $\widehat{\mathbf{D}}$, which is defined also only by a metric $\mathbf{g}$ in a metric compatible form, when $\widehat{T}^i_{\ jk} = 0$ and $\widehat{T}^a_{\ bc} = 0$ but $\widehat{T}^i_{\ ja}, \widehat{T}^a_{\ ji}$ and $\widehat{T}^a_{\ bi}$ are not zero, see (12). The coefficients $\widehat{\Gamma}^\gamma_{\ \alpha\beta} = (\widehat{L}^i_{\ jk}, \widehat{L}^a_{\ bk}, \widehat{C}^i_{\ jc}, \widehat{C}^a_{\ bc})$ of the canonical d-connection, with respect to the N-adapted frames, are:

$$\widehat{L}^i_{\ jk} = \frac{1}{2} g^{ir}(e_k g_{jr} + e_j g_{kr} - e_r g_{jk}),$$

$$\widehat{L}^a_{\ bk} = e_b(N^a_k) + \frac{1}{2} h^{ac}(e_k h_{bc} - h_{dc} e_b N^d_k - h_{db} e_c N^d_k), \qquad (18)$$

$$\widehat{C}^i_{\ jc} = \frac{1}{2} g^{ik} e_c g_{jk}, \quad \widehat{C}^a_{\ bc} = \frac{1}{2} h^{ad}(e_c h_{bd} + e_c h_{cd} - e_d h_{bc}).$$

The nontrivial N-adapted coefficients for torsion, curvature and Ricci d-tensors, i.e $\widehat{\mathbf{T}}^\alpha_{\ \beta\gamma}$, $\widehat{\mathbf{R}}^\alpha_{\ \beta\gamma\delta}$ and $\widehat{\mathbf{R}}_{\alpha\beta}$, can be computed in explicit form my introducing the coefficients (18) into respective formulas (12), (16) and (17).

We note that any geometric construction for the canonical d-connection $\widehat{\mathbf{D}}$ can be redefined by the Levi Civita connection by using the formula

$$_\lfloor\Gamma^\gamma_{\ \alpha\beta} = \widehat{\mathbf{\Gamma}}^\gamma_{\ \alpha\beta} + {}_\lfloor Z^\gamma_{\ \alpha\beta}, \qquad (19)$$

where the both connections $_\lfloor\Gamma^\gamma_{\ \alpha\beta}, \widehat{\Gamma}^\gamma_{\ \alpha\beta}$ and the distortion tensor $_\lfloor Z^\gamma_{\ \alpha\beta}$ can be defined by the generic off-diagonal metric (9), or (equivalently) by d-metric (8) and the coefficients of N-connection (1).[5] If we work with nonholonomic constraints on the dynamics/geometry of

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[5]See, for instance, Refs. [10–12], for explicit formulas expressing $_\lfloor Z^\gamma_{\ \alpha\beta}$ through the components g_{ij}, h_{ab}, N^a_i, their respective inverse values and their partial derivatives.

gravity fields, it is more convenient to use a N-adapted approach. For other purposes, it is preferred to use only the Levi Civita connection. Introducing the distortion relation (19) into respective formulas (12), (16) and (17) written for $\widehat{\mathbf{\Gamma}}^{\gamma}_{\alpha\beta}$, we get the deformation relations of type

$$
\begin{aligned}
{}_{|}T^{\alpha}_{\ \beta\gamma} &= \widehat{\mathbf{T}}^{\alpha}_{\ \beta\gamma} + {}^{T}_{|}Z^{\gamma}_{\ \alpha\beta} = 0, \\
{}_{|}R^{\alpha}_{\ \beta\gamma\delta} &= \widehat{\mathbf{R}}^{\alpha}_{\ \beta\gamma\delta} + {}^{R}_{|}\widehat{\mathbf{Z}}^{\alpha}_{\ \beta\gamma\delta}, \qquad {}_{|}R_{\ \beta\gamma} = \widehat{\mathbf{R}}_{\ \beta\gamma} + {}^{Ric}_{|}\widehat{\mathbf{Z}}_{\ \beta\gamma},
\end{aligned}
\tag{20}
$$

where ${}_{|}R_{\ \beta\gamma} = {}_{|}R_{\ \gamma\beta}$ but $\widehat{\mathbf{R}}_{\ \beta\gamma} \neq \widehat{\mathbf{R}}_{\ \gamma\beta}$ and Z-values can be computed by an explicit deformation calculus for respective tensors.

Finally, we conclude that prescribing a nonintegrable splitting by a nonholonomic distribution, or a N-adapted frame structure, on a (pseudo) Riemannian manifold one can model the geometry of this manifold in two equivalent forms, both defined by the same metric structure (9): the first one is the standard approach with the Levi Civita connection, resulting in nonzero torsion and symmetric Ricci tensor, and the second one is the N-adapted approach, with induced torsion (by the off-diagonal terms of the metric (9)) and nonsymmetric Ricci tensor for the canonical d-connection.

2.2 Nonholonomic Ricci Flows and Nonsymmetric Metrics

The normalized holonomic Ricci flows on a real parameter $\chi \in [0, \chi_0)$, for symmetric metrics with respect to the coordinate base $\partial_{\underline{\alpha}} = \partial/\partial u^{\underline{\alpha}}$, are described by the equations

$$
\frac{\partial}{\partial\chi} g_{\underline{\alpha}\underline{\beta}} = -2 \, {}_{|}R_{\underline{\alpha}\underline{\beta}} + \frac{2r}{5} g_{\underline{\alpha}\underline{\beta}},
\tag{21}
$$

where the normalizing factor $r = \int {}_{|}R \, dV / dV$, with the Ricci scalar ${}_{|}R = g^{\underline{\alpha}\underline{\beta}} \, {}_{|}R_{\underline{\alpha}\underline{\beta}}$ is defined by the metric structure $g_{\underline{\alpha}\underline{\beta}}$ and Levi Civita connection ∇, is introduced in order to preserve the volume V. For N-anholonomic Ricci flows, the coefficients $g_{\underline{\alpha}\underline{\beta}}$ are parametrized in the form (9). Heuristic arguments for postulating such equations, similarly to the Einstein equations, are discussed in Refs. [2, 3, 6–9] and, for nonholonomic manifolds, [10, 11].

The Ricci flow equations (21) can be written in equivalent form by distinguishing the N-connection coefficients, but preserving the Ricci tensor defined by the Levi Civita connection,

$$
\frac{\partial}{\partial\chi} g_{ij} = 2\left[N_i^a N_j^b \left({}_{|}\underline{R}_{ab} - \frac{r}{5} h_{ab} \right) - {}_{|}\underline{R}_{ij} + \frac{r}{5} g_{ij} \right] - h_{cd} \frac{\partial}{\partial\chi} (N_i^c N_j^d),
\tag{22}
$$

$$
\frac{\partial}{\partial\chi} h_{ab} = -2\left({}_{|}\underline{R}_{ab} - \frac{r}{5} h_{ab} \right),
\tag{23}
$$

$$
\frac{\partial}{\partial\chi} (N_j^e h_{ae}) = -2\left({}_{|}\underline{R}_{ia} - \frac{r}{5} N_j^e h_{ae} \right),
\tag{24}
$$

where the coefficients are defined with respect to local coordinate basis.[6]

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[6]We underline some indices or symbols for geometric objects if we wont to emphasize that they are defined with respect to a coordinate basis.

With respect to N-adapted frames, the nonholonomic Ricci flows for the canonical d-connection $\widehat{\mathbf{D}}$ when some off-diagonal metric coefficients can be nonsymmetric are defined by equations

$$\frac{\partial}{\partial \chi} g_{ij} = -2\widehat{R}_{ij} + \frac{2r}{5} g_{ij} - h_{cd}\frac{\partial}{\partial \chi}(N_i^c N_j^d), \tag{25}$$

$$\frac{\partial}{\partial \chi} h_{ab} = -2\widehat{R}_{ab} + \frac{2r}{5} h_{ab}, \tag{26}$$

$$\frac{\partial}{\partial \chi} \breve{g}_{ia} = \widehat{R}_{ia}, \qquad \frac{\partial}{\partial \chi} \breve{g}_{ai} = \widehat{R}_{ai}, \tag{27}$$

where $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$ with respect to N-adapted basis (4), $y^3 = v$ and χ can be, for instance, the time like coordinate, $\chi = t$, or any parameter or extra dimension coordinate. It should be emphasized that there are three important differences between the system of (22)–(24) and (25)–(27):

1. The first system is for connection ∇ but the second one is for $\widehat{\mathbf{D}}$.
2. Because, in general, $\widehat{R}_{ia} \neq \widehat{R}_{ai}$, see formulas (17) for $\widehat{\mathbf{D}}$, even $\widehat{\mathbf{R}}_{\alpha\beta}$ is stated to be defined by a symmetric (9), equivalently by a symmetric (8), we must extend the metric to contain nonsymmetric coefficients of type (10), when $\breve{g}_{ib} = g_{ib} + a_{ib}$ and $\breve{g}_{bi} = g_{bi} + a_{bi}$, where $g_{ib} = g_{bi}$ and $a_{ib} = -a_{bi}$, and (27) transform into

$$\frac{\partial}{\partial \chi} g_{ia} = \widehat{R}_{(ia)}, \qquad \frac{\partial}{\partial \chi} a_{bi} = \widehat{R}_{[bi]}, \tag{28}$$

   where $\widehat{R}_{ia} = \widehat{R}_{(ia)} + \widehat{R}_{[bi]}$ is the decomposition of this d-tensor into symmetric and antisymmetric parts. In Refs. [10, 11],[7] we restricted our considerations only for N-anholonomic configurations with $\widehat{R}_{ia}(\chi) = 0$ when the Ricci flows transforms symmetric metrics only into symmetric ones. From (28), one follows that we get nontrivial antisymmetric values $a_{bi}(\chi)$ even if $\frac{\partial}{\partial \chi} g_{ia} = 0$ for $\widehat{R}_{(ia)} = \widehat{R}_{[bi]} = 0$. It is easy to prove this with respect to a coordinate basis when (28) transform into

$$\frac{\partial}{\partial \chi}(N_j^e \, \underline{a}_{be}) = 0, \tag{29}$$

   where $\underline{a}_{be}$ are coordinate coefficients of a_{bi} formally written with respect to N-adapted basis (compare with equations (24) for the Levi Civita connection, redefined in N-adapted form for the canonical d-connection). The equation (29) have nontrivial solutions for $N_j^e(\chi)$ and $\underline{a}_{be}(\chi)$ with nontrivial $_1 R_{\alpha\beta}$, but with $\widehat{R}_{(ia)} = \widehat{R}_{[bi]} = 0$, see deformation formulas (20).
3. The system of equations for N-adapted Ricci flows (25)–(27) must be completed with a system of equations for the N-adapted frames (5),

$$\mathbf{e}_\alpha(\chi) = \mathbf{e}_\alpha^{\underline{\alpha}}(\chi, u)\partial_{\underline{\alpha}}$$

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[7]We note that the system of denotations for the nonsymmetric metrics in this work are elaborated in a different form in order to try to elaborate in our further works a unified approach to nonholonomic geometries both with symmetric and nonsymmetric metrics.

defined by the coefficients

$$\mathbf{e}_{\underline{\alpha}}^{\alpha}(\chi, u) = \begin{bmatrix} e_i^{\underline{i}}(\chi, u) & N_i^b(\chi, u)\, e_b^{\underline{a}}(\chi, u) \\ 0 & e_a^{\underline{a}}(\chi, u) \end{bmatrix},$$

with

$$g_{ij}(\chi, u) = e_i^{\underline{i}}(\chi, u)\, e_j^{\underline{j}}(\chi, u)\eta_{\underline{ij}} \text{ and } h_{ab}(\chi, u) = e_a^{\underline{a}}(\chi, u)\, e_b^{\underline{b}}(\chi, u)\eta_{\underline{ab}},$$

where $\eta_{\underline{ij}} = \mathrm{diag}[\pm 1, \ldots \pm 1]$ and $\eta_{\underline{ab}} = \mathrm{diag}[\pm 1, \ldots \pm 1]$ establish the signature of $g_{\underline{\alpha\beta}}(u)$, is given by equations

$$\frac{\partial}{\partial \chi}\mathbf{e}_{\underline{\alpha}}^{\alpha} = \mathbf{g}^{\alpha\beta}\widehat{\mathbf{R}}_{\beta\underline{\gamma}}\,\mathbf{e}_{\alpha}^{\underline{\gamma}},$$

see details in Refs. [10, 11]. Here we note that Ricci flows of N-adapted frames are defined by the equations

$$\frac{\partial}{\partial \chi}e_{\underline{\alpha}}^{\alpha} = g^{\alpha\underline{\beta}}\,_{\shortmid}R_{\underline{\beta\gamma}}\,e_{\alpha}^{\gamma}$$

if we define the Ricci flow equations in non N-adapted form just only for the Levi Civita connection ∇.

In further sections, we shall develop a geometric method of constructing exact solutions for the system of Ricci flow evolution equations (25), (26) and (29) defining N-adapted transforms of symmetric metrics into nonsymmetric ones. We shall also present explicit examples when physically valuable exact solutions in general relativity evolve under such nonholonomic flows into respective nonsymmetric metrics.

3 An Ansatz for Constructing Nonsymmetric Ricci Flow Solutions

We consider a four dimensional (4D) manifold $\mathbf{V}$ of necessary smooth class and conventional splitting of dimensions $\dim \mathbf{V} = n + m$ for $n = 2$ and $m = 2$. The local coordinates are labeled in the form $u^\alpha = (x^i, y^a) = (x^i, y^3 = v, y^4 = y)$, for $i = 1, 2$ and $a, b, \ldots = 3, 4$. Any coordinates from a set u^α can be a three dimensional (3D) space or time like variable when Ricci flows of geometric objects will be parametrized by a real χ.

3.1 Off-Diagonal Ansatz for Einstein Spaces and Ricci Flows

We consider an ansatz of type (8) parametrized in the form

$$\begin{aligned}
\mathbf{g} &= g_1(x^1, x^2)dx^1 \otimes dx^1 + g_2(x^1, x^2)dx^2 \otimes dx^2 \\
&\quad + h_3(x^k, v)\,\delta v \otimes \delta v + h_4(x^k, v)\,\delta y \otimes \delta y, \\
\delta v &= dv + w_i(x^k, v)dx^i, \quad \delta y = dy + n_i(x^k, v)dx^i
\end{aligned} \tag{30}$$

with the coefficients defined by some necessary smooth class functions

$$g_{1,2} = g_{1,2}(x^1, x^2), \quad h_{3,4} = h_{3,4}(x^i, v), \quad w_i = w_i(x^k, v), \quad n_i = n_i(x^k, v).$$

The off-diagonal terms of this metric, written with respect to the coordinate dual frame $du^\alpha = (dx^i, dy^a)$, can be redefined to state a N-connection structure $\mathbf{N} = [N_i^3 = w_i(x^k, v),$ $N_i^4 = n_i(x^k, v)]$ with a N-elongated co-frame (4) parametrized as

$$e^1 = dx^1, \quad e^2 = dx^2, \quad \mathbf{e}^3 = \delta v = dv + w_i dx^i, \quad \mathbf{e}^4 = \delta y = dy + n_i dx^i. \quad (31)$$

This coframe is dual to the local basis

$$\mathbf{e}_i = \frac{\partial}{\partial x^i} - w_i(x^k, v)\frac{\partial}{\partial v} - n_i(x^k, v)\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial v}, \quad e_4 = \frac{\partial}{\partial y}. \quad (32)$$

We emphasize that the metric (30) does not depend on variable y, i.e. it posses a Killing vector $e_4 = \partial/\partial y$, and distinguishes the dependence on the so-called "anisotropic" variable $y^3 = v$.

In order to model Ricci flows, we have to consider dependencies on flow parameter of the metric coefficients,

$$\begin{aligned}
{}^\chi \mathbf{g} = \mathbf{g}(\chi) &= g_1(x^k, \chi)dx^1 \otimes dx^1 + g_2(x^k, \chi)dx^2 \otimes dx^2 \\
&\quad + h_3(x^k, v, \chi) \, {}^\chi\delta v \otimes {}^\chi\delta v + h_4(x^k, v, \chi) \, {}^\chi\delta y \otimes {}^\chi\delta y, \\
{}^\chi\delta v &= dv + w_i(x^k, v, \chi)dx^i, \quad {}^\chi\delta y = dy + n_i(x^k, v, \chi)dx^i
\end{aligned} \quad (33)$$

with corresponding flows for N-adapted bases,

$$\begin{aligned}
\mathbf{e}_\alpha = (\mathbf{e}_i, e_a) &\to {}^\chi\mathbf{e}_\alpha = ({}^\chi\mathbf{e}_i, e_a) = \mathbf{e}_\alpha(\chi) = (\mathbf{e}_i(\chi), e_a), \\
\mathbf{e}^\alpha = (e^i, \mathbf{e}^a) &\to {}^\chi\mathbf{e}^\alpha = (e^i, {}^\chi\mathbf{e}^a) = \mathbf{e}^\alpha(\chi) = (e^i, \mathbf{e}^a(\chi))
\end{aligned}$$

defined by $w_i(x^k, v) \to w_i(x^k, v, \lambda)$, $n_i(x^k, v) \to n_i(x^k, v, \lambda)$ in (32), (31).

Computing the components of the Ricci and Einstein tensors for the metric (33) (see details on similar calculus in Refs. [10–12]), one proves that the corresponding family of Ricci tensors for the canonical d-connection with respect to N-adapted frames are compatible with the sources (they can be any matter fields, string corrections, Ricci flow parameter derivatives of metric, ...)

$$\mathbf{\Upsilon}_\beta^\alpha = [\Upsilon_1^1 = \Upsilon_2^2 = \Upsilon_2(x^k, v, \chi), \quad \Upsilon_3^3 = \Upsilon_4^4 = \Upsilon_4(x^k, \chi)]. \quad (34)$$

For simplicity, in this work, we shall analyze Ricci flows of the so-called nonholonomic Einstein spaces defined by solutions of equations

$$\begin{aligned}
\widehat{R}^i{}_j &= {}^h\lambda(x^i, \chi)\delta_j^i, \quad \widehat{R}^a{}_b = {}^v\lambda(x^i, v, \chi)\delta_b^a, \\
\widehat{R}_{3i} &= \widehat{R}_{i3} = 0, \quad \widehat{R}_{4i} = \widehat{R}_{i4} = 0,
\end{aligned} \quad (35)$$

where ${}^h\lambda(x^i, \chi)$ and ${}^v\lambda(x^i, v, \chi)$ state an effective polarized cosmological constant (in our case, they are nonhomogeneous and anisotropic dependencies on coordinates) which can be computed for certain models of gravity with quantum corrections, higher order contributions and so on.

The equations (35) for the ansatz (33) with any fixed value of χ, i.e. for the ansatz (30), transform into this system of partial differential equations:

$$\widehat{R}_1^1 = \widehat{R}_2^2(\chi)$$

$$= \frac{1}{2g_1 g_2}\left[\frac{g_1^\bullet g_2^\bullet}{2g_1} + \frac{(g_2^\bullet)^2}{2g_2} - g_2^{\bullet\bullet} + \frac{g_1' g_2'}{2g_2} + \frac{(g_1')^2}{2g_1} - g_1''\right] = {}^h\lambda(x^i, \chi), \qquad (36)$$

$$\widehat{R}_3^3 = \widehat{R}_4^4(\chi) = \frac{1}{2h_3 h_4}\left[h_4^*\left(\ln\sqrt{|h_3 h_4|}\right)^* - h_4^{**}\right] = {}^v\lambda(x^i, v, \chi), \qquad (37)$$

$$\widehat{R}_{3i} = -w_i(\chi)\frac{\beta(\chi)}{2h_4(\chi)} - \frac{\alpha_i(\chi)}{2h_4(\chi)} = 0, \qquad (38)$$

$$\widehat{R}_{4i} = -\frac{h_4(\chi)}{2h_3(\chi)}\left[n_i^{**}(\chi) + \gamma(\chi)n_i^*(\chi)\right] = 0, \qquad (39)$$

where, for $h_{3,4}^* \neq 0$,

$$\alpha_i(\chi) = h_4^*(\chi)\partial_i\phi(\chi), \quad \beta(\chi) = h_4^*(\chi)\,\phi^*(\chi), \qquad (40)$$

$$\gamma(\chi) = \frac{3h_4^*(\chi)}{2h_4(\chi)} - \frac{h_3^*(\chi)}{h_3(\chi)}, \quad \phi(\chi) = \ln\left|\frac{h_4^*(\chi)}{\sqrt{|h_3(\chi)h_4(\chi)|}}\right|, \qquad (41)$$

when the necessary partial derivatives are written in the form $a^\bullet = \partial a/\partial x^1$, $a' = \partial a/\partial x^2$, $a^* = \partial a/\partial v$. We note that the off-diagonal gravitational interactions and Ricci flows can model locally anisotropic configurations even if $\lambda_2 = \lambda_4$, or both values vanish.

Summarizing the results for (30) with arbitrary signatures $\epsilon_\alpha = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$, where $\epsilon_\alpha = \pm 1$ and $h_3^* \neq 0$ and $h_4^* \neq 0$, one proves, see details in [10–12], that any off-diagonal metric

$$\begin{aligned}
{}^\circ\mathbf{g} = {} &\epsilon_1 g_1(x^i)\,dx^1 \otimes dx^1 + \epsilon_2 g_2(x^i)\,dx^2 \otimes dx^2 \\
&+ \epsilon_3 h_0^2(x^i)[f^*(x^i, v)]^2|\varsigma(x^i, v)|\,\delta v \otimes \delta v \\
&+ \epsilon_4[f(x^i, v) - f_0(x^i)]^2\,\delta y^4 \otimes \delta y^4, \\
\delta v = {} &dv + w_k(x^i, v)dx^k, \quad \delta y^4 = dy^4 + n_k(x^i, v)dx^k, \qquad (42)
\end{aligned}$$

with the coefficients being of necessary smooth class and the indices with "hat" running the values $i, j, \ldots = 1, 2$, where $g_k(x^i)$ is a solution of the 2D equation (36) for a given source $\Upsilon_4(x^i)$,

$$\varsigma(x^i, v) = \varsigma_{[0]}(x^i) + \frac{\epsilon_4}{8}h_0^2(x^i)\int {}^v\lambda(x^k, v)f^*(x^i, v)[f(x^i, v) - f_0(x^i)]dv,$$

and the N-connection coefficients $N_i^3 = w_i(x^k, v)$, $N_i^4 = n_i(x^k, v)$ are computed following the formulas

$$w_i = -\frac{\partial_i \varsigma(x^k, v)}{\varsigma^*(x^k, v)}, \qquad (43)$$

$$n_k = {}^1 n_k(x^i) + {}^2 n_k(x^i)\int \frac{[f^*(x^i, v)]^2 \varsigma(x^i, v)}{[f(x^i, v) - f_0(x^i)]^3}dv, \qquad (44)$$

define respectively exact solutions of the Einstein equations (38) and (39). It should be emphasized that such solutions depend on arbitrary functions $f(x^i, v)$, for $f^* \neq 0$, $f_0(x^i)$,

$h_0^2(x^i)$, $\varsigma_{[0]}(x^i)$, $^1n_k(x^i)$, $^2n_k(x^i)$ and $^v\lambda(x^{\widehat{k}}, v)$, $^h\lambda(x^{\widehat{i}})$. Such values for the corresponding signatures $\epsilon_\alpha = \pm 1$ have to be stated by certain boundary conditions following some physical considerations. Here we note that this class of solutions of Einstein equations with nonholonomic variable depend on integration functions. It is more general than those for diagonal ansatz depending, for instance, on one radial like variable like in the case of the Schwarzschild solution (when the Einstein equations are reduced to an effective nonlinear ordinary differential equation, ODE). In the case of ODE, the integral varieties depend on integration constants to be defined from certain boundary/asymptotic and symmetry conditions, for instance, from the constraint that far away from the horizon the Schwarzschild metric contains corrections from the Newton potential. Because our ansatz (30) transforms (35) in a system of nonlinear partial differential equations transforms, the solutions depend not only on integration constants but also on certain classes of integration functions.

The ansatz of type (30) with $h_3^* = 0$ but $h_4^* \neq 0$ (or, inversely, $h_3^* \neq 0$ but $h_4^* = 0$) consist more special cases and request a bit different methods for constructing exact solutions.

3.2 Solutions for Ricci Flows and Nonsymmetric Metrics

For families of solutions parametrized by χ, we consider flows of the generating functions, $g_1(x^i, \chi)$, or $g_2(x^i, \chi)$, and $f(x^i, v, \chi)$, and various types of integration functions and sources, for instance, $n_{k[1]}(x^i, \chi)$ and $n_{k[2]}(x^i, \chi)$ and $\Upsilon_2(x^k, v, \chi)$, respectively, in formulas (43) and (44). Let us analyze an example of exact solutions of (25), (26) and (29) defined by an ansatz with nontrivial nonsymmetric component for the metric parametrized in the form $\underline{a}_{bc}(x^i, \chi)$.

We search a class of solutions when

$$g_1 = \epsilon_1\varpi(x^i, \chi), \quad g_2 = \epsilon_2\varpi(x^i, \chi), \quad \varpi(x^i, \chi) = \exp\{2\psi(x^i, \chi)\},$$

$$h_3 = h_3(x^i, v), \quad h_4 = h_4(x^i, v), \quad \underline{a}_{34} = \underline{a}_{34}(x^i, \chi)$$

for a family of ansatz (33) with any prescribed signatures $\epsilon_\alpha = \pm 1$ and non-negative functions ϖ and h. The equations (29) results into

$$\partial_\chi(w_2\underline{a}_{34}) = 0 \quad \text{and} \quad \partial_\chi(n_1\underline{a}_{34}) = 0. \tag{45}$$

Following a tensor calculus, adapted to the N-connection, for the canonical d-connection, we express the integral variety for a class of nonholonomic Ricci flows as

$$\epsilon_1(\ln|\varpi|)^{\bullet\bullet} + \epsilon_2(\ln|\varpi|)^{''} = 2\,^v\lambda - h_4\partial_\chi(n_2)^2, \tag{46}$$

$$h_3 = h\varsigma_3$$

for

$$\varsigma_3(x^i, v) = \varsigma_{3[0]}(x^i) - \frac{1}{4}\int \frac{^v\lambda h h_4}{h_4^*}dv,$$

$$\sqrt{|h|} = h_{[0]}(x^i)\left(\sqrt{|h_4(x^i, v)|}\right)^* \tag{47}$$

and, for $\varphi = -\ln|\sqrt{|h_3h_4|}/|h_5^*||$,

$$w_1 = (\varphi^*)^{-1}\varphi^\bullet, w_2 = (\varphi^*)^{-1}\varphi', \tag{48}$$

$$n_1 = n_2 = \,^1n(x^i, \chi) + \,^2n(x^i, \chi)\int dv\, h_3/\left(\sqrt{|h_4|}\right)^3,$$

where the partial derivatives are denoted in the form $\varphi^\bullet = \partial\varphi/\partial x^1$, $\varphi' = \partial\varphi/\partial x^2$, $\varphi^* = \partial\varphi/\partial v$, $\partial_\chi = \partial/\partial\chi$, and arbitrary h_4 when $h_4^* \neq 0$. For $\lambda = 0$, we shall consider $\varsigma_{3[0]} = 1$ and $h_{[0]}(x^i) = \text{const}$ in order to solve the vacuum Einstein equations. There is a class of solutions when

$$h_4 \int dv\, h_3/\left(\sqrt{|h_4|}\right)^3 = C(x^i),$$

for a function $C(x^i)$. This is compatible with the condition (47) and we can chose such configurations, for instance, with $^1 n = 0$ and any $^2 n(x^i, \chi)$ and $\varpi(x^i, \chi)$ solving (46).

Putting together (46)–(48), we get a class of solutions of the system (36)–(39) for nonholonomic Ricci flows of metrics of type (33),

$$
\begin{aligned}
^\chi \mathbf{g} = {}& \varpi(x^i, \chi)\left[\epsilon_1 dx^1 \otimes dx^1 + \epsilon_2 dx^2 \otimes dx^2\right] \\
& + h_3(x^i, v)\,\delta v \otimes \delta v + h_4(x^i, v)\ ^\chi \delta y \otimes\ ^\chi \delta y, \\
\delta v = {}& dv + w_1(x^i, v)dx^1 + w_2(x^i, v)dx^2, \\
^\chi \delta y = {}& dy + n_1(x^i, v, \chi)[dx^1 + dx^2].
\end{aligned}
\tag{49}
$$

Such solutions describe in general form the Ricci flows of nonholonomic Einstein spaces constrained to relate in a mutually compatible form the evolution of horizontal part of metric, $\varpi(x^i, \chi)$, with the evolution of N-connection coefficients $n_1 = n_2 = n_1(x^i, v, \chi)$. We have to impose certain boundary/ initial conditions for $\chi = 0$, beginning with an explicit solution of the Einstein equations, in order to define the integration functions and state an evolution scenario for such classes of metrics and connections.

The family of metrics (49) defines Ricci flows of N-anholonomic Einstein spaces constructed for the canonical d-connection. We can extract solutions for the Levi Civita connection if we constrain the coefficients of such metrics to satisfy the conditions:

$$
\begin{aligned}
\epsilon_1 \psi^{\bullet\bullet}(x^k, \chi) + \epsilon_2 \psi''(x^k, \chi) &= -\,^h\lambda(x^k, \chi), \\
\frac{h_4^*(x^i, v)\phi(x^i, v)}{h_3(x^i, v)h_4(x^i, v)} &= -\,^v\lambda(x^i, v), \\
w_2(x^i, v)w_1^*(x^i, v) - w_1(x^i, v)w_2^*(x^i, v) &= w_2^\bullet(x^i, v) - w_1'(x^i, v), \\
n_1'(x^k, \chi) - n_2^\bullet(x^k, \chi) &= 0,
\end{aligned}
\tag{50}
$$

where $\varpi = e^{\psi(x^k, \chi)}$, $n_i = n_i(x^k, \chi)$, $w_i = \partial_i \phi/\phi^*$, see (41).[8]

We can extend the class of metrics (33) to nontrivial nonsymmetric configurations with

$$^\chi \breve{\mathbf{g}} = {}^\chi \mathbf{g} + {}^\chi \mathbf{a}$$

when $^\chi \mathbf{a} = a_{34}(x^i, v, \chi)dv \wedge dy$ is constrained to satisfy the conditions (45). Here we note that constructing exact solutions with generic off-diagonal metrics and nonholonomic variables for Ricci flows in Refs. [28–32] we took the trivial solution with $a_{34} = 0$. In this paper, $a_{34}(x^i, v, \chi)$ can be arbitrary functions solving the nonholonomic Ricci flow equations. It can be nonzero, even we started with a symmetric metric configuration but the N-connection

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[8]Proofs of such conditions are given, for instance, in Refs. [31, 32].

structure naturally generates a nonsymmetric metric component. Such metrics are not constrained to satisfy the field equations in a model of nonsymmetric gravity like in Refs. [15, 16, 23–25]. For the ansatz considered in this section, we can consider a week decomposition around $^x\mathbf{g}$ when $^x\mathbf{a}$ is also constrained to satisfy the corresponding system of gravitational field equations, for certain values of χ. A comprehensive study of Ricci flows of solutions of nonsymmetric gravity is a topic for further our investigations.

4 pp-Wave Ricci Flows of Taub-NUT Metrics into Nonsymmetric Metrics

The anholonomic frame method can be applied in order to generate Ricci flow solutions for various classes 4D metrics [29, 32]. In this section, we examine how nonholonomic Ricci flows of a Taub-NUT metric may result in nonsymmetric configurations if the flow parameter is associated to a time like coordinate for pp-waves.

We consider a 'primary' ansatz written in a form similar to (8)

$$
\begin{aligned}
\tilde{\mathbf{g}} = {}& \tilde{g}_1(x^k, v, y^4)(dx^1)^2 + \tilde{g}_2(x^k, v, y^4)(dx^2)^2 \\
& + \tilde{h}_3(x^k, v, y^4)(\check{b}^3)^2 + \tilde{h}_4(x^k, v, y^4)(\check{b}^4)^2,
\end{aligned}
\tag{51}
$$

$$
\tilde{b}^3 = dv + \tilde{w}_i(x^k, v, y^4)\, dx^i, \quad \tilde{b}^4 = dy^4 + \tilde{n}_i(x^k, v, y^4)\, dx^i,
$$

following the parametrizations

$$
x^1 = r, x^2 = \vartheta, y^3 = v = p, y^4 = \varphi
$$

$$
\tilde{g}_1(r) = F^{-1}(r), \tilde{g}_2(r) = (r^2 + n^2),
$$

$$
\tilde{h}_3(r) = -F(r), \tilde{h}_4(r, \vartheta) = (r^2 + n^2)a(\vartheta),
$$

$$
\tilde{w}_1(\vartheta) = -2nw(\vartheta), \tilde{w}_2 = 0, \tilde{n}_i = 0,
$$

where the functions and coordinates are those for the quadratic element

$$
d\tilde{s}^2 = F^{-1}dr^2 + (r^2 + n^2)d\vartheta^2 - F(r)[dt - 2nw(\vartheta)d\varphi]^2 + (r^2 + n^2)a(\vartheta)d\varphi^2
\tag{52}
$$

defining the topological Taub-NUT-AdS/dS spacetimes [35–37] with NUT charge n. The function $F(r)$ takes three different values,

$$
F(r) = \frac{r^4 + (\varepsilon l^2 + n^2)r^2 - 2\mu r l^2 + \varepsilon n^2(l^2 - 3n^2) + (1 - |\varepsilon|)n^2}{l^2(n^2 + r^2)}
$$

for $\varepsilon = 1, 0, -1$, defining respectively

$$
\left\{
\begin{array}{l}
U(1) \text{ fibrations over } S^2; \\
U(1) \text{ fibrations over } T^2; \\
U(1) \text{ fibrations over } H^2;
\end{array}
\right.
\quad \text{for} \quad
\left\{
\begin{array}{c}
a(\vartheta) = \sin^2\vartheta, w(\vartheta) = \cos\vartheta, \\
a(\vartheta) = 1, w(\vartheta) = \vartheta, \\
a(\vartheta) = \sinh^2\vartheta, w(\vartheta) = \cosh\vartheta.
\end{array}
\right.
$$

The ansatz (52) for $\varepsilon = 1, 0, -1$ but $n = 0$ recovers correspondingly the spherical, toroidal and hyperbolic Schwarzschild-AdS/dS solutions of 4D Einstein equations with cosmological constant $\lambda = -3/l^2$ and mass parameter μ. The metrics (51) and (52) are related by coordinate transform $(r, \vartheta, t, \varphi) \to (r, \vartheta, p(\vartheta, t, \varphi), \varphi)$ with a new time like coordinate p when

$$
dt - 2nw(\vartheta)d\varphi = dp - 2nw(\vartheta)d\vartheta,
$$

and $t \rightarrow p$ are substituted in (52) for

$$t \rightarrow p = t - \int \nu^{-1}(\vartheta, \varphi) d\xi(\vartheta, \varphi)$$

with

$$d\xi = -\nu(\vartheta, \varphi) d(p - t) = \partial_\vartheta \xi \, d\vartheta + \partial_\varphi \xi d\varphi,$$

when

$$d(p - t) = 2nw(\vartheta)(d\vartheta - d\varphi).$$

The last formulas state that the functions $\nu(\vartheta, \varphi)$ and $\xi(\vartheta, \varphi)$ are taken to solve the equations

$$\partial_\vartheta \xi = -2nw(\vartheta)\nu \quad \text{and} \quad \partial_\varphi \xi = 2nw(\vartheta)\nu.$$

For instance, the solutions of such equations are generated by

$$\xi = e^{f(\varphi - \vartheta)} \quad \text{and} \quad \nu = \frac{1}{2nw(\vartheta)} \frac{df}{dx} e^{f(\varphi - \vartheta)}$$

for $x = \varphi - \vartheta$.

We perform an anholonomic transform $\check{\mathbf{N}} \rightarrow N$ and $\check{\mathbf{g}} = (\check{g}, \check{h}) \rightarrow \mathbf{g} = (g, h)$, when

$$
\begin{aligned}
g_1 &= \eta_1(r, \vartheta)\tilde{g}_1(r), \quad g_2 = \eta_2(r, \vartheta)\tilde{g}_2(r), \\
h_3 &= \eta_3(r, \vartheta, p)\tilde{h}_3(r), \quad h_4 = \eta_4(r, \vartheta, p)\tilde{h}_4(r, \vartheta), \\
w_1 &= \eta_1^3(r, \vartheta, p)\tilde{w}_1(\vartheta), \quad w_2 = w_2(r, \vartheta, p), \\
n_1 &= n_1(r, \vartheta, p), \quad n_2 = n_2(r, \vartheta, p).
\end{aligned}
\tag{53}
$$

This results in the "target" metric ansatz

$$
\begin{aligned}
\mathbf{g} &= g_1(r, \vartheta)(dr)^2 + g_2(r, \vartheta)(d\vartheta)^2 + h_3(r, \vartheta, p)(b^3)^2 + h_4(r, \vartheta, p)(b^4)^2, \\
b^3 &= dp + w_1(r, \vartheta, p) \, dr + w_2(r, \vartheta, p) \, d\vartheta, \\
b^4 &= d\varphi + n_1(r, \vartheta, p) \, dr + n_2(r, \vartheta, p) \, d\vartheta.
\end{aligned}
\tag{54}
$$

Our aim is to state the coefficients when this off-diagonal metric ansatz defines solutions of the nonholonomic Ricci flow equations (25), (26) and (29) for $\chi = p$. We shall construct a family of exact solutions of the system of equations with polarized cosmological constants (36)–(39) following the same steps used for deriving formulas (46)–(48) for the metric (49). By a corresponding 2D coordinate transform $x^i \rightarrow x^{\tilde{i}}(r, \vartheta)$, the horizontal component of the family of metrics (54) can be always diagonalized and represented in conformally flat form,

$$g_1(r, \vartheta)(dr)^2 + g_2(r, \vartheta)(d\vartheta)^2 = e^{2\psi(x^{\tilde{i}})}\left[\epsilon_1(dx^{\tilde{1}})^2 + \epsilon_2(dx^{\tilde{2}})^2\right],$$

where the values $\epsilon_i = \pm 1$ depend on chosen signature and $\psi(x^{\tilde{i}})$ is a solution of

$$\epsilon_1 \psi^{\bullet\bullet} + \epsilon_2 \psi'' = {}^h\lambda(x^{\tilde{i}}).$$

For other metric coefficients, one obtains the relations

$$\phi(r, \vartheta, p) = \ln \left| h_4^* / \sqrt{|h_3 h_4|} \right|,$$

for

$$(e^\phi)^* = -2\lambda_{[v]}(r, \vartheta, p)\sqrt{|h_3 h_4|},$$

$$|h_3| = 4e^{-2\phi(r,\vartheta,p)} \left[\left(\sqrt{|h_4|} \right)^* \right]^2, \quad |h_4^*| = -(e^\phi)^* / 4 \, {}^v\lambda.$$

It is convenient to represent such solutions in the form

$$h_4 = \epsilon_4 [b(r, \vartheta, p) - b_0(r, \vartheta)]^2, \quad h_3 = 4\epsilon_3 e^{-2\phi(r,\vartheta,p)} [b^*(r, \vartheta, p)]^2, \tag{55}$$

where $\epsilon_a = \pm 1$ depend on fixed signature, $b_0(r, \vartheta)$ and $\phi(r, \vartheta, p)$ can be arbitrary functions and $b(r, \vartheta, p)$ is any function with b^* related to ϕ and ${}^v\lambda$.

The N-connection coefficients are of type

$$n_k = {}^1 n_k(r, \vartheta) + {}^2 n_k(r, \vartheta) \, \hat{n}_k(r, \vartheta, p),$$

where

$$\hat{n}_k(r, \vartheta, p) = \int h_3 (\sqrt{|h_4|})^{-3} dp,$$

and ${}^1 n_k(r, \vartheta)$ and ${}^2 n_k(r, \vartheta)$ are integration functions and $h_4^* \neq 0$.

The above constructed coefficients for the metric and N-connection depend on arbitrary integration functions. We have to constrain such integral varieties in order to construct Ricci flow solutions with the Levi Civita connection, see similar details in Sect. 3 of Ref. [29]. One considers a matrix equation for matrices $\widetilde{g}(r, \vartheta) = [2 \, {}^h\lambda(r, \vartheta) \, g_{ij}(r, \vartheta)]$ and $\widetilde{w}(r, \vartheta, p) = [w_i(r, \vartheta, p) \, w_j(r, \vartheta, p)]$

$$\widetilde{g}(r, \vartheta) = h_3(r, \vartheta, p) \frac{\partial}{\partial p} \widetilde{w}(r, \vartheta, p). \tag{56}$$

This equation can be compatible for such 2D systems of coordinates when $\widetilde{g}$ is not diagonal because $\widetilde{w}$ is also not diagonal. For 2D subspaces, the coordinate and frame transforms are equivalent but such configurations should be correspondingly adapted to the nonholonomic structure defined by $\widetilde{w}(r, \vartheta, p)$ which is possible for a general 2D coordinate system. One introduces the transforms

$$g_{ij} = e_i^{i'}(x^{k'}(r, \vartheta)) e_j^{j'}(x^{k'}(r, \vartheta)) g_{i'j'}(x^{k'})$$

and

$$w_{i'}(x^{k'}) = e_{i'}^i(x^{k'}(r, \vartheta)) w_i((r, \vartheta, p))$$

associated to a coordinate transform $(r, \vartheta) \rightarrow x^{k'}(r, \vartheta))$ with $g_{i'j'}(x^{k'})$ defining, in general, a symmetric but non-diagonal (2×2)-dimensional matrix.

The equation (56) can be integrated in explicit form by separation of variables in ϕ, b, h_3 and $w_{i'}$, when

$$\phi = \widehat{\phi}(x^{i'})\check{\phi}(p), h_3 = \widehat{h_3}(x^{i'})\check{h}_3(p),$$

$$w_{i'} = \widehat{w}_{i'}(x^{k'})q(p), \text{ for } \widehat{w}_{i'} = -\partial_{i'} \ln |\widehat{\phi}(x^{k'})|, q = (\partial_p \check{\phi}(p))^{-1}$$

where separation of variables for h_3 is related to a similar separation of variables $b = \widehat{b}(x^{i'})\check{b}(p)$ as follows from (55). We get the matrix equation

$$\widetilde{g}(x^{k'}) = \alpha_0 \widehat{h}_3(x^{i'})\widetilde{w}_0(x^{k'}),$$

where the matrix $\widetilde{w}_0$ has components $(\widehat{w}_{i'}\widehat{w}_{k'})$ and constant $\alpha_0 \neq 0$ is chosen from any prescribed relation

$$\check{h}_3(p) = \alpha_0 \partial_p \left[\partial_p \check{\phi}(p)\right]^{-2}. \tag{57}$$

We conclude that any given functions $\widehat{\phi}(x^{k'}), \check{\phi}(p)$ and $\widehat{h}_3(x^{i'})$ and constant α_0 we can generate solutions of the Ricci flow equations (25) and (26) for $n_i = 0$ with the metric coefficients parametrized in the same form as for the solution of the Einstein equations (36)–(39). In a particular case, we can take $\check{\phi}(p)$ to be a periodic or solitonic type function.

The last step in constructing flow solutions is to solve (26) for the ansatz (54) redefined for coordinates $x^{k'} = x^{k'}(r, \vartheta)$,

$$\frac{\partial}{\partial p}h_a = 2 \, {}^{v}\lambda(x^{k'}, p) \, h_a.$$

This equation is compatible if $h_4 = \varsigma(x^{k'})h_3$ for any prescribed function $\varsigma(x^{k'})$. We can satisfy this condition by corresponding parametrizations of function $\phi = \widehat{\phi}(x^{i'}) \check{\phi}(p)$ and/or $b = \widehat{b}(x^{i'})\check{b}(p)$, see (55). As a result, we can compute the effective cosmological constant for such Ricci flows,

$$\lambda_{[v]}(x^{k'}, p) = \partial_p \ln |h_3(x^{k'}, p)|,$$

which for solutions of type (57) is defined by a polarization running in time,

$$\lambda_{[v]}(p) = \alpha_0 \partial_p^2 [\partial_p \check{\phi}(p)]^{-2}.$$

In this case, we can identify α_0 with a cosmological constant $\lambda = -3/l^2$, for primary Taub-NUT configurations, if we choose such $\check{\phi}(p)$ that $\partial_p^2 [\partial_p \check{\phi}(p)]^{-2} \to 1$ for $p \to 0$.

Putting together the coefficients of metric and N-connection with the formulas constructed above, one obtains a family of symmetric metrics

$$\mathbf{g} = \alpha_0 \widehat{h}_3(x^{i'}) \left\{ \partial_{i'} \ln |\widehat{\phi}(x^{k'})| \, \partial_{j'} \ln |\widehat{\phi}(x^{k'})| \, dx^{i'} dx^{j'} + \partial_p [\partial_p \check{\phi}(p)]^{-2} \right.$$

$$\times \left[[dp - (\partial_p \check{\phi}(p))^{-1}(dx^{i'} \partial_{i'} \ln |\widehat{\phi}(x^{k'})|)]^2 + \varsigma(x^{k'})(d\varphi)^2 \right] \right\}. \tag{58}$$

The nontrivial w-coefficients, $w_{i'} = -(\partial_p \check{\phi}(p))^{-1} \partial_{i'} \ln |\widehat{\phi}(x^{k'})|$, induce a nontrivial solution of the equations for the nonsymmetric component of the metric, see (45), which for the symmetric configuration (58) is written in the form

$$\partial_p(w_{2'}\underline{a}_{34}) = 0.$$

The solution of this equation can be represented in the form

$$\underline{a}_{34} = (\partial_p \check{\phi}(p))\underline{a}_{34}^{[0]}(x^{i'}),$$

where $\underline{a}_{34}^{[0]}(x^{i'})$ is to be defined from certain boundary conditions for a fixed system of coordinates $x^{i'}$.

The general nonsymmetric off-diagonal metric defining the pp-wave like Ricci wave evolution of 4D Taub NUT spaces is

$$\check{\mathbf{g}} = \mathbf{g} + \mathbf{a} = \alpha_0 \, \widehat{h}_3(x^{i'}) \, \big\{ \partial_{i'} \ln |\widehat{\phi}(x^{k'})| \, \partial_{j'} \ln |\widehat{\phi}(x^{k'})| \, dx^{i'} dx^{j'}$$
$$+ \partial_p [\partial_p \check{\phi}(p)] \, \big[[dp - (\partial_p \check{\phi}(p))^{-1} (dx^{i'} \partial_{i'} \ln |\widehat{\phi}(x^{k'})|)]^2 + \varsigma(x^{k'})(d\varphi)^2 \big] \big\}$$
$$+ (\partial_p \check{\phi}(p)) \underline{a}_{34}^{[0]}(x^{i'}) \, dp \wedge d\varphi. \tag{59}$$

This metric ansatz depends on certain type of arbitrary integration and generation functions $\widehat{h}_3(x^{i'})$, $\widehat{\phi}(x^{k'})$, $\varsigma(x^{k'})$, $\check{\phi}(p)$ and $\underline{a}_{34}^{[0]}(x^{i'})$ and on a constant α_0 which can be identified with the primary cosmological constant. It was derived by considering nonholonomic deformations of some classes of 4D Taub-NUT solutions parametrized by the primary metric (51) by considering polarizations functions (53) deforming the coefficients of the primary metrics into the target ones for corresponding Ricci flows. The target metric (58) model 4D Einstein spaces with "horizontally" polarized, $^h\lambda(x^{k'})$ and "vertically" running, $^v\lambda(p)$, cosmological constant managed by the Ricci flow solutions. If we suppose that there is a nonsymmetric tensor with nontrivial components $\underline{a}_{34}^{[0]}(x^{i'})$ in a spacetime region, we can perform scenaria with nontrivial nonsymmetric Ricci flow evolution of metrics.

We conclude that if the primary 4D topological Taub-NUT-AdS/dS spacetimes have the structure of $U(1)$ fibrations over 2D hypersurfaces (spheres, toruses or hyperboloids) than their nonholonomic deformations to Ricci flow solutions with effectively polarized/running cosmological constant define certain classes of generalized 4D Einstein spaces as foliations on the corresponding 2D hypersurfaces. This holds true if the nonholonomic structures are chosen to be integrable and for the Levi Civita connection. Additionally, such foliations may be enabled with pp-wave moving nonsymmetric components for metrics.

Finally, we note that in more general cases, with nontrivial torsion, for instance, induced from other models of classical or quantum gravity, we deal with "nonintegrable" foliated structures, i.e. with nonholonomic Riemann–Cartan manifolds provided with effective nonlinear connection structure induces by off-diagonal metric terms. The nonsymmetric components of the metrics under nonholonomic Ricci flow evolutions of Riemann–Cartan structures can be constructed in a similar form.

5 Solitonic pp-Waves and Nonsymmetric Ricci Flows of Schwarzschild Metrics

Alternatively to the solutions constructed in previous section, one can be generated new classes of solutions of nonholonomic Ricci flow equations when the evolution parameter is not identified to a spacetime coordinate. From physical point of view, we may treat such solutions to define gravity models with variable on χ constants (in general, being effectively polarized by holonomic–nonholonomic variables) and generalized (non) symmetric metrics and metric compatible affine connections adapted to the nonlinear connection structure. The aim of this section is to construct and analyze three classes of Ricci flow evolution equations deforming nonholonomically certain physically valuable exact solutions in general relativity into geometric configurations with nonsymmetric metric.

5.1 Solitonic pp-waves in vacuum Einstein gravity and Ricci flows

We show how the anholonomic frame method can be applied for generating 4D metrics with nontrivial antisymmetric terms defined by nonlinear pp-waves and solitonic interactions for vanishing sources and the Levi Civita connection.

We use an ansatz of type (33),

$$
\delta s^2_{[4]} = -e^{\psi(x,y,\chi)}(dx^2 + dy^2)
$$

$$
- 2\kappa(x,y,p)\,\eta_3(x,y,p)\delta p^2 + \frac{\eta_4(x,y,p)}{8\kappa(x,y,p)}\delta v^2 \tag{60}
$$

$$
\delta p = dp + w_2(x,y,p)dx + w_3(x,y,p)dy,
$$

$$
\delta v = dv + n_2(x,y,p,\chi)dx + n_3(x,y,p,\chi)dy
$$

where the local coordinates are labelled $x^1 = x,\ x^2 = y,\ y^3 = p,\ y^4 = v,$ and the nontrivial metric coefficients are parametrized

$$
\check{g}_1 = -1, \quad \check{g}_2 = -1, \quad \check{h}_3 = -2\kappa(x,y,p), \quad \check{h}_4 = 1/\,8\kappa(x,y,p),
$$

$$
g_\alpha = \eta_\alpha \check{g}_\alpha.
$$

For trivial polarizations $\eta_\alpha = 1$ and $w_{2,3} = 0,\ n_{2,3} = 0$, the metric (60) is just the pp-wave solution of vacuum Einstein equations [38], i.e.

$$
\delta s^2_{[4pp]} = \epsilon_1\, d\varkappa^2 - dx^2 - dy^2 - 2\kappa(x,y,p)\, dp^2 + dv^2/8\kappa(x,y,p), \tag{61}
$$

for any $\kappa(x,y,p)$ solving

$$
\kappa_{xx} + \kappa_{yy} = 0,
$$

with $p = z + t$ and $v = z - t$, where (x,y,z) are usual Cartesian coordinates and t is the time like coordinate. The simplest explicit examples of such solutions are given by

$$
\kappa = (x^2 - y^2)\sin p,
$$

defining a plane monochromatic wave, or

$$
\kappa = \frac{xy}{(x^2 + y^2)^2 \exp[p_0^2 - p^2]}, \quad \text{for } |p| < p_0;
$$

$$
= 0, \quad \text{for } |p| \geq p_0,
$$

defining a wave packet travelling with unit velocity in the negative z direction.

For an ansatz of type (60), we write

$$
\eta_4 = 5\kappa b^2 \quad \text{and} \quad \eta_3 = h_0^2(b^*)^2/2\kappa.
$$

A 3D solitonic solution of Einstein equations and its Ricci flows can be generated if b is subjected to the condition to solve a solitonic equation. For instance, we can take $\eta_4 = \eta(x,y,p)$ for the solitonic equation

$$
\eta^{\bullet\bullet} + \epsilon(\eta' + 6\eta\,\eta^* + \eta^{***})^* = 0, \quad \epsilon = \pm 1, \tag{62}
$$

or other nonlinear wave configuration. As a simple example, we can chose a parametrization when

$$b(x, y, p) = \check{b}(x, y)q(p)k(p),$$

for any $\check{b}(x, y)$ and any pp-wave $\kappa(x, y, p) = \check{\kappa}(x, y)k(p)$, where $q(p) = 4\tan^{-1}(e^{\pm p})$ is the solution of "one dimensional" solitonic equation

$$q^{**} = \sin q. \tag{63}$$

In this case,

$$w_1 = [(\ln|qk|)^*]^{-1}\partial_x \ln|\check{b}| \quad \text{and} \quad w_2 = [(\ln|qk|)^*]^{-1}\partial_y \ln|\check{b}|. \tag{64}$$

The final step in constructing such vacuum Einstein solutions is to chose any two functions $n_{1,2}(x, y)$ satisfying the conditions $n_1^* = n_2^* = 0$ and $n_1' - n_2^\bullet = 0$ which are necessary for Riemann foliated structures with the Levi Civita connection, see conditions (50). This means that in the integrals of type (48) we shall fix the integration functions ${}^2n_{1,2} = 0$ but take such ${}^1n_{1,2}(x, y)$ satisfying $({}^1n_1)' - ({}^1n_2)^\bullet = 0$.

Summarizing the results, for vanishing source (vanishing effective cosmological constants) in (36), (37) and (50), and for a fixed value of χ, we obtain the 4D vacuum off-diagonal metric

$$\delta s_{[4offd]}^2 = -(dx^2 + dy^2) - h_0^2\check{b}^2[(qk)^*]^2\delta p^2 + \check{b}^2(qk)^2\delta v^2,$$

$$\delta p = dp + [(\ln|qk|)^*]^{-1}\partial_x \ln|\check{b}|\, dx + [(\ln|qk|)^*]^{-1}\partial_y \ln|\check{b}|\, dy,$$

$$\delta v = dv + {}^1n_1 dx + {}^1n_2 dy, \tag{65}$$

defining nonlinear gravitational interactions of a pp-wave $\kappa = \check{\kappa}k$ and a soliton q, depending on certain type of integration functions and constants stated above. Such vacuum Einstein metrics can be generated in a similar form for 3D or 2D solitons but the constructions will be more cumbersome and for non-explicit functions, see construction and discussion of a number of similar solutions in Ref. [34].

At the next step, we generalize the ansatz (65) in a form describing normalized Ricci flows of the mentioned type vacuum solutions extended for a prescribed constant ${}^0\lambda = r/5$ necessary for normalization. We chose

$$\delta s_{[\chi]}^2 = -(dx^2 + dy^2) - h_0^2\check{b}^2(\chi)[(qk)^*]^2\delta p^2 + \check{b}^2(\chi)(qk)^2\delta v^2,$$

$$\delta p = dp + [(\ln|qk|)^*]^{-1}\partial_x \ln|\check{b}|\, dx + [(\ln|qk|)^*]^{-1}\partial_y \ln|\check{b}|\, dy,$$

$$\delta v = dv + {}^1n_1(\chi)dx + {}^1n_2(\chi)dy, \tag{66}$$

where we introduced the parametric dependence on χ,

$$b(x, y, p, \chi) = \check{b}(x, y, \chi)q(p)k(p).$$

The values $\check{b}^2(\chi)$ and ${}^1n_2(\chi)$ are constrained to be solutions of

$$\frac{\partial}{\partial \chi}\left[\check{b}^2({}^1n_{1,2})^2\right] = -2\,{}^0\lambda \quad \text{and} \quad \frac{\partial}{\partial \chi}\check{b}^2 = 2\,{}^0\lambda\check{b}^2 \tag{67}$$

in order to solve, respectively, (36) and (37) with evolution on χ. As a matter of principle, we can consider a flow dependence as a factor $\psi(\chi)$ before $(dx^2 + dy^2)$. For simplicity, we have chosen a minimal extension of vacuum Einstein solutions in order to describe non-holonomic flows of the v-components of metrics adapted to the flows of N-connection coefficients ${}^1n_{1,2}(\chi)$. Such nonholonomic constraints on metric coefficients define Ricci flows of families of vacuum Einstein solutions defined by nonlinear interactions of a 3D soliton and a pp-wave.

Putting the values $w_i(x, y, p)$, defined by formulas (64), and ${}^1n_{1,2}(x, y, \chi)$, defined by formulas (67), into (29), see also (45), we get the equations for nonsymmetric component of metrics, $\underline{a}_{34}(x, y, p, \chi)$, under Ricci flows

$$\partial_\chi (w_2 \underline{a}_{34}) = 0 \quad \text{and} \quad \partial_\chi ({}^1n_1 \underline{a}_{34}) = 0. \tag{68}$$

There are two classes of solutions of this system of evolution equations: They first class is given by the conditions

$$w_2 \neq 0, \underline{a}_{34} \neq 0, \quad \partial_\chi (\underline{a}_{34}) = 0 \quad \text{and} \quad \partial_\chi ({}^1n_1) = 0,$$

which means that a nontrivial value of $\underline{a}_{34}$ will not evolve under Ricci flows and not interact with the solitonic pp-waves from the symmetric part of the metric. The second class of solutions, more interesting from physical point of view (with evolution on χ derived for corresponding configurations of solitonic pp-waves), can be constructed if the function $\check{b} = \check{b}(x, \chi)$ does not depend on variable y. In this case, $w_2 = 0$, but $w_1 \neq 0$, see (64), which allows solutions with nontrivial ${}^1n_{1,2}(x, y, \chi)$ and $\underline{a}_{34}(x, y, p, \chi)$ subjected to conditions

$$\partial_\chi ({}^1n_1 \underline{a}_{34}) = 0. \tag{69}$$

The resulting families of metrics with nontrivial nonsymmetric components defining a solitonic pp-wave evolution of the primary pp-wave symmetric vacuum solution can be parametrized in the form

$$\begin{aligned}
\check{\mathbf{g}} = \mathbf{g} + \mathbf{a} = &- (dx \otimes dx + dy \otimes dy) \\
&- [h_0 \check{b}(x, \chi)(q(p)k(p))^*]^2 \delta p \otimes \delta p + [\check{b}(x, \chi)(q(p)k(p)]^2 \delta v \otimes \delta v \\
&+ \underline{a}_{34}(x, y, p, \chi) dp \wedge dv, \tag{70}
\end{aligned}$$
$$\delta p = dp + [(\ln|q(p)k(p)|)^*]^{-1} \partial_x \ln|\check{b}(x, \chi)| \, dx,$$
$$\delta v = dv + {}^1n_1(x, y, \chi) dx + {}^1n_2(x, y, \chi) dy,$$

where $q(p) = 4\tan^{-1}(e^{\pm p})$, for any ${}^1n_1$ and ${}^1n_2$ with $({}^1n_1)' - ({}^1n_2)^\bullet = 0$ and, for instance, $k(p) = \sin p$, or $= 1/\exp[p_0^2 - p^2]$; $h_0 = \text{const}$ and $p_0 = \text{const}$, for any functions $\check{b}(x, \chi)$ and $\underline{a}_{34}(x, y, p, \chi)$ satisfying the conditions (67) and (69). The evolution in (70) is on a real parameter χ which is different from the class of solutions in (59) where the evolution parameter was fixed to be a time like coordinate. It should be noted that we took a very special case of parametrization of pp-wave and solitonic interactions and their evolution in order to be able to describe in explicit form such nonlinear Ricci flow configurations. As a matter of principle, such configurations can be defined in nonexplicit form for more general types of solitonic pp-wave interactions. We conclude that normalized nonholonomic Ricci flows of vacuum pp-wave vacuum Einstein solutions naturally evolve into metrics with nonsymmetric components.

5.2 Nonholonomic Ricci Flows and 4D (Non) Symmetric Deformations of Stationary
    Backgrounds

We show that Ricci flows subjected to corresponding nonholonomic deformations of the
Schwarzschild metric result in nonsymmetric metrics. There are analyzed such evolutions
defined by generic off-diagonal flows and interactions with solitonic pp-waves. We develop
for spaces with nonsymmetric metrics the methods developed in Refs. [31, 32]. We non-
holonomically deform on angular variable φ the Schwarzschild type solution into a generic
off-diagonal stationary metric.

5.2.1 General Nonholonomic Deformations

The primary quadratic element is taken

$$\delta s_{[1]}^2 = -d\xi^2 - r^2(\xi)\,d\vartheta^2 - r^2(\xi)\sin^2\vartheta\,d\varphi^2 + \varpi^2(\xi)\,dt^2, \tag{71}$$

where the local coordinates and nontrivial metric coefficients are parametrized in the form

$$x^1 = \xi, \quad x^2 = \vartheta, \quad y^3 = \varphi, \quad y^4 = t,$$
$$\check{g}_1 = -1, \quad \check{g}_2 = -r^2(\xi), \quad \check{h}_3 = -r^2(\xi)\sin^2\vartheta, \quad \check{h}_4 = \varpi^2(\xi), \tag{72}$$

for

$$\xi = \int dr \left|1 - \frac{2\mu}{r} + \frac{\varepsilon}{r^2}\right|^{1/2} \quad \text{and} \quad \varpi^2(r) = 1 - \frac{2\mu}{r} + \frac{\varepsilon}{r^2}.$$

For the constants $\varepsilon \to 0$ and μ being a point mass, the element (71) defines the Schwarz-
schild solution written in spacetime spherical coordinates $(r, \vartheta, \varphi, t)$.[9]

By nonholonomic deformations, $g_i = \eta_i \check{g}_i$ and $h_a = \eta_a \check{h}_a$, where $(\check{g}_i, \check{h}_a)$ are given by
data (72), we get an ansatz for which the coefficients are constrained to define nonholonomic
Einstein spaces,

$$\delta s_{[1def]}^2 = -\eta_1(\xi)d\xi^2 - \eta_2(\xi)r^2(\xi)\,d\vartheta^2$$
$$\qquad - \eta_3(\xi, \vartheta, \varphi)r^2(\xi)\sin^2\vartheta\,\delta\varphi^2 + \eta_4(\xi, \vartheta, \varphi)\varpi^2(\xi)\,\delta t^2,$$
$$\delta\varphi = d\varphi + w_1(\xi, \vartheta, \varphi)d\xi + w_2(\xi, \vartheta, \varphi)d\vartheta,$$
$$\delta t = dt + n_1(\xi, \vartheta)d\xi + n_2(\xi, \vartheta)d\vartheta, \tag{73}$$

where there are used 3D spacial spherical coordinates, $(\xi(r), \vartheta, \varphi)$ or (r, ϑ, φ). This class of
metrics is of type (42), with coordinates $x^1 = \xi, x^2 = \vartheta, y^3 = \varphi, y^4 = t$.

The equation (37) for zero source gives this relation for the horizontal coefficients of
symmetric metric and respective polarization functions:

$$-h_0^2(b^*)^2 = \eta_3(\xi, \vartheta, \varphi)r^2(\xi)\sin^2\vartheta \quad \text{and} \quad b^2 = \eta_4(\xi, \vartheta, \varphi)\varpi^2(\xi),$$

---

[9]For simplicity, in this work, we shall consider only the case of vacuum solutions, not analyzing a more
general possibility when $\varepsilon = e^2$ is related to the electric charge for the Reissner–Nordström metric (see, for
example, [39]). In our further considerations, we shall treat ε as a small parameter, for instance, defining a
small deformation of a circle into an ellipse (eccentricity).

for

$$|\eta_3| = (h_0)^2 |\check{h}_4/\check{h}_3| \left[\left(\sqrt{|\eta_4|}\right)^*\right]^2, \tag{74}$$

with $h_0 = \text{const}$, where $\check{h}_a$ are stated by the Schwarzschild solution for the chosen system of coordinates and η_4 can be any function satisfying the condition $\eta_4^* \neq 0$. We can compute the polarizations η_1 and η_2, when $\eta_1 = \eta_2 r^2 = e^{\psi(\xi,\vartheta,\chi)}$, from (36) with zero source, written in the form

$$\psi^{\bullet\bullet} + \psi'' = 0.$$

The solutions of (38) and (39) for vacuum configurations of the Levi Civita connection are given by

$$w_1 = \partial_\xi(\sqrt{|\eta_4|}\varpi)/\left(\sqrt{|\eta_4|}\right)^* \varpi, \quad w_2 = \partial_\vartheta(\sqrt{|\eta_4|})/\left(\sqrt{|\eta_4|}\right)^*$$

and any $n_{1,2} = {}^1 n_{1,2}(\xi,\vartheta)$ for which ${}^1 n_1' - {}^1 n_2^{\bullet} = 0$.

Putting the defined values of the coefficients in the ansatz (73) we find a class of exact vacuum solutions of the Einstein equations defining stationary nonholonomic deformations of the Schwarzschild metric,

$$\delta s_{[1]}^2 = -e^{\psi}(d\xi^2 + d\vartheta^2) - h_0^2 \left[\left(\sqrt{|\eta_4|}\right)^*\right]^2 \varpi^2 \, \delta\varphi^2 + \eta_4 \varpi^2 \, \delta t^2,$$

$$\delta\varphi = d\varphi + \frac{\partial_\xi(\sqrt{|\eta_4|}\varpi)}{\left(\sqrt{|\eta_4|}\right)^* \varpi} d\xi + \frac{\partial_\vartheta(\sqrt{|\eta_4|})}{\left(\sqrt{|\eta_4|}\right)^*} d\vartheta,$$

$$\delta t = dt + {}^1 n_1 d\xi + {}^1 n_2 d\vartheta, \tag{75}$$

where, at this step, the coefficients do not depend on Ricci flow parameter χ. Such vacuum solutions were constructed to transform nonholonomically a static black hole solution into Einstein spaces with locally anisotropic backgrounds (on coordinate φ) defined by an arbitrary function $\eta_4(\xi,\vartheta,\varphi)$ with $\partial_\varphi \eta_4 \neq 0$, an arbitrary $\psi(\xi,\vartheta)$ solving the 2D Laplace equation and certain integration functions ${}^1 n_{1,2}(\xi,\vartheta)$ and integration constant h_0^2. In general, the solutions from the target set of metrics do not define black holes and do not describe obvious physical situations. Nevertheless, they preserve the singular character of the coefficient ϖ^2 vanishing on the horizon of a Schwarzschild black hole if we take only smooth integration functions. We can also consider a prescribed physical situation when, for instance, η_4 mimics 3D, or 2D, solitonic polarizations on coordinates ξ, ϑ, φ, or on ξ, φ.

In this section, we consider a different model of nonholonomic Ricci flow evolution when only the N-connection coefficients depend on flow parameter χ, but the d-metric coefficients are re-scaled in the form: $g_{ij} \to e^{-{}^0\lambda\chi} g_{ij}$ and $h_{ab} \to e^{-{}^0\lambda\chi} h_{ab}$, where g_{ij} and h_{ab} are stationary values given by off-diagonal solution (75). The "nearest" extension to flows of N-connection coefficients

$$w_1 \to w_1(\chi) = \eta_1^3(\xi,\vartheta,\varphi,\chi) \frac{\partial_\xi(\sqrt{|\eta_4|}\varpi)}{\left(\sqrt{|\eta_4|}\right)^* \varpi},$$

$$w_2 \to w_2(\chi) = \eta_2^3(\xi,\vartheta,\varphi,\chi) \frac{\partial_\vartheta(\sqrt{|\eta_4|})}{\left(\sqrt{|\eta_4|}\right)^*},$$

$$n_1 \to n_1(\chi) = \eta_1^4(\xi, \vartheta, \chi)\,{}^1n_1,$$
$$n_2 \to n_2(\chi) = \eta_2^4(\xi, \vartheta, \chi)\,{}^1n_2, \tag{76}$$

for

$$n_1'(\chi) - n_2^\bullet(\chi) = 0 \quad \text{and} \quad \eta_i^a(\xi, \vartheta, \chi) \to 1 \quad \text{for } \chi \to 0. \tag{77}$$

For ${}^0\lambda = 2r/5$ and $R_{\alpha\beta} = 0$, (25) is satisfied if

$$h_0^2 \left[\left(\sqrt{|\eta_4|}\right)^*\right]^2 \frac{\partial (w_i)^2}{\partial \chi} = \eta_4 \frac{\partial (n_i)^2}{\partial \chi}. \tag{78}$$

We can represent the integral of these equations in the form:

$$(w_i)^2 = (n_i)^2 \frac{\eta_4}{h_0^2 \left[\left(\sqrt{|\eta_4|}\right)^*\right]^2} + F_i, \tag{79}$$

where $F_i(\xi, \vartheta, \varphi,)$ are integration functions. The symmetric metric coefficients for such Ricci flows are proportional to those for the exact solutions for vacuum nonholonomic deformations but rescaled and with respect to evolving N-adapted dual basis

$$\delta\varphi(\chi) = d\varphi + w_2(\xi, \vartheta, \varphi, \chi)d\xi + w_3(\xi, \vartheta, \varphi, \chi)d\vartheta, \tag{80}$$
$$\delta t = dt + n_2(\xi, \varphi, \chi)d\xi + n_3(\xi, \vartheta, \chi)d\vartheta,$$

with the coefficients being defined by any solution of (78).

The nontrivial coefficient of the nonsymmetric metric can be computed by integrating (29), which, in this section, reduce to

$$\partial_\chi(w_2\underline{a}_{34}) = 0 \quad \text{and} \quad \partial_\chi(n_1\underline{a}_{34}) = 0,$$

with the partial derivatives on χ of N-connection coefficients constrained to satisfy (78). We can express the equations for $\underline{a}_{34} = \underline{a}_{34}(\xi, \vartheta, \varphi, \chi)$ in the form

$$\partial_\chi(\eta_i^4(\xi, \vartheta, \chi)\underline{a}_{34}) = 0, \tag{81}$$

where $n_i = \eta_i^4\,{}^1n_i$ are subjected to the conditions (77) and the coefficients w_i are computed following formulas (79).

We obtain that the family of nonsymmetric metrics

$$\check{\mathbf{g}} = \mathbf{g} + \mathbf{a} = -e^{-\,{}^0\lambda\chi+\psi}(d\xi \otimes d\xi + d\vartheta \otimes d\vartheta)$$
$$- h_0^2\, e^{-\,{}^0\lambda\chi}\left[\left(\sqrt{|\eta_4|}\right)^*\right]^2 \varpi^2\, \delta\varphi \otimes \delta\varphi$$
$$+ e^{-\,{}^0\lambda\chi}\,\eta_4\varpi^2\, \delta t \otimes \delta t + \underline{a}_{34}\, d\varphi \wedge dt,$$
$$\delta\varphi = d\varphi + \eta_1^3 \frac{\partial_\xi(\sqrt{|\eta_4|}\varpi)}{(\sqrt{|\eta_4|})^*\varpi}d\xi + \eta_2^3 \frac{\partial_\vartheta(\sqrt{|\eta_4|})}{(\sqrt{|\eta_4|})^*}d\vartheta,$$
$$\delta t = dt + \eta_1^4 \times\, {}^1n_1 d\xi + \eta_2^4 \times\, {}^1n_2 d\vartheta, \tag{82}$$

with the coefficients constrained to satisfy the conditions (77)–(78), (81) and (79) define the Ricci flow evolution of a Schwarzschild metric when the flows are considered for the N-connection coefficients.

5.2.2 Solutions with Small Nonholonomic Polarizations

The class of solutions (82) in defined in a very general form. Let us extract a subclasses of solutions related to the Schwarzschild metric. We consider decompositions on a small parameter $0 < \varepsilon < 1$ in (75), when

$$\sqrt{|\eta_3|} = q_3^{\hat{0}}(\xi, \varphi, \vartheta) + \varepsilon q_3^{\hat{1}}(\xi, \varphi, \vartheta) + \varepsilon^2 q_3^{\hat{2}}(\xi, \varphi, \vartheta) \ldots ,$$

$$\sqrt{|\eta_4|} = 1 + \varepsilon q_4^{\hat{1}}(\xi, \varphi, \vartheta) + \varepsilon^2 q_4^{\hat{2}}(\xi, \varphi, \vartheta) \ldots ,$$

where the "hat" indices label the coefficients multiplied to $\varepsilon, \varepsilon^2, \ldots$. The conditions (74) are expressed in the form

$$\varepsilon h_0 \sqrt{\left|\frac{\check{h}_4}{\check{h}_3}\right|} \left(q_4^{\hat{1}}\right)^* = q_3^{\hat{0}}, \quad \varepsilon^2 h_0 \sqrt{\left|\frac{\check{h}_4}{\check{h}_3}\right|} \left(q_4^{\hat{2}}\right)^* = \varepsilon q_3^{\hat{1}}, \ldots$$

We take the integration constant, for instance, to satisfy the condition $\varepsilon h_0 = 1$ (choosing a corresponding system of coordinates). For such small deformations, we prescribe a function $q_3^{\hat{0}}$ and define $q_4^{\hat{1}}$, integrating on φ (or inversely, prescribing $q_4^{\hat{1}}$, then taking the partial derivative ∂_φ, to compute $q_3^{\hat{0}}$). In a similar form, there are related the coefficients $q_3^{\hat{1}}$ and $q_3^{\hat{2}}$. An important physical situation arises when we select the conditions when such small nonholonomic deformations define rotoid configurations. This is possible, for instance, if

$$2q_4^{\hat{1}} = \frac{q_0(r)}{4\mu^2} \sin(\omega_0\varphi + \varphi_0) - \frac{1}{r^2}, \tag{83}$$

where ω_0 and φ_0 are constants and the function $q_0(r)$ has to be defined by fixing certain boundary conditions for polarizations. In this case, the coefficient before δt^2 is

$$\eta_4 \varpi^2 = 1 - \frac{2\mu}{r} + \varepsilon \left(\frac{1}{r^2} + 2q_4^{\hat{1}}\right).$$

This coefficient vanishes and defines a small deformation of the Schwarzschild spherical horizon into a an ellipsoidal one (rotoid configuration) given by

$$r_+ \simeq \frac{2\mu}{1 + \varepsilon \frac{q_0(r)}{4\mu^2} \sin(\omega_0\varphi + \varphi_0)}.$$

Such solutions with ellipsoid symmetry seem to define static black ellipsoids (they were investigated in details in Refs. [40, 41]). The ellipsoid configurations were proven to be stable under perturbations and transform into the Schwarzschild solution far away from the ellipsoidal horizon. In general relativity, this class of vacuum metrics violates the conditions of black hole uniqueness theorems [39] because the "surface" gravity is not constant for stationary black ellipsoid deformations. Nonholonomic Ricci flows generalize the theory to nonsymmetric metrics (similar effects can be modelled by string and/or noncommutative gravity corrections [34, 42] but with different parameters and without nonsymmetric components of metrics).

 We can construct an infinite number of ellipsoidal locally anisotropic black hole deformations. Nevertheless, they present physical interest because they preserve the spherical

topology, have the Minkowski asymptotic and the deformations can be associated to certain classes of geometric spacetime distorsions related to generic off-diagonal metric terms. Putting $\varphi_0 = 0$, in the limit $\omega_0 \to 0$, we get $q_5^{\hat{1}} \to 0$ in (83). This allows to state the limits $q_3^{\hat{0}} \to 1$ for $\varepsilon \to 0$ in order to have a smooth limit to the Schwarzschild solution for $\varepsilon \to 0$. Here, one must be emphasized that to extract the spherical static black hole solution is possible if we parametrize for $\chi = 0$

$$\delta\varphi = d\varphi + \varepsilon \frac{\partial_\xi (\sqrt{|\eta_4|}\varpi)}{\left(\sqrt{|\eta_4|}\right)^* \varpi} d\xi + \varepsilon \frac{\partial_\vartheta (\sqrt{|\eta_4|})}{\left(\sqrt{|\eta_4|}\right)^*} d\vartheta$$

and

$$\delta t = dt + \varepsilon n_2(\xi, \vartheta) d\xi + \varepsilon n_3(\xi, \vartheta) d\vartheta.$$

For Ricci flows on N-connection coefficients, such stationary rotoid configurations evolve with respect to small deformations of co-frames (80), $\delta\varphi(\chi)$ and $\delta t(\chi)$, with the coefficients proportional to ε.

One can be defined certain more special cases when $q_4^{\hat{2}}$ and $q_3^{\hat{1}}$ (as a consequence) are of solitonic locally anisotropic nature. In result, such solutions will define small stationary deformations of the Schwarzschild solution embedded into a background polarized by anisotropic solitonic waves.

6 Conclusions and Perspectives

In this paper we have considered nonholonomic Ricci flows of (pseudo) Riemannian metrics resulting in solutions of evolution equations containing nonsymmetric components of metrics. We have seen that a variety of well-known physically valuable solutions in general relativity (like Taub NUT, pp-wave and solitonic wave and Schwarzschild metrics) will get nontrivial anti-symmetric components of metrics if their physical parameters and/or certain components of metric are allowed to run on a Ricci flow parameter which can be identified with a time-like coordinate (for one type of solutions) or considered to be a general real one varying on a finite interval (for the second type of solutions). A generic property of such constructions is that certain classes of diagonal metrics (they can be, or not, exact solutions) are extended to generic off-diagonal ones which define exact solutions for nonhomogeneous and locally anisotropic Einstein spaces (with effective cosmological constant polarized on coordinate and/or time variables; in a particular case, we can consider a usual cosmological constant vanishing for vacuum configurations).

The off-diagonal metric coefficients can be effectively transformed into coefficients of a nonholonomic frame with associated nonlinear connection (N-connection) structures. Such geometric methods were elaborated in generalized Lagrange and Finsler geometry, but we emphasize that in this work we restrict our considerations only to primary (pseudo) Riemannian spaces and Riemann–Cartan spaces with effective torsion induced by nonholonomic frames. The existence of nontrivial off-diagonal/N-connection coefficients is crucial for obtaining in result of the Ricci flow evolution of nonsymmetric components of (target) metrics. Moreover, we have found a possible relation to the Dirac's hypothesis of variation of physical constants which in our approach can be explained by running of such constants under Ricci flows, together with possible locally anisotropic polarizations and more general evolutions into nonholonomically deformed to (non) symmetric metrics and generalized

connection structures. The characterization of target metrics in relation to generalized gravity and matter field equations remain to be found. For simplicity, in this work we restricted our analysis only to nonsymmetric metrics induced by Ricci flows and not as solutions of certain field dynamics and constraints equations.

The results obtained in this paper, together with the former study of the nonholonomic Ricci flow evolution of gravitational and regular mechanical systems, provide a strong geometric ground for theories with nonsymmetric metrics. If we relax the hidden condition that (Ricci) flows of Riemannian metrics must result only in Riemann metrics and subject the evolution scenaria to certain nonholonomic constraints, we get that all "exotic" geometries with symmetric and nonsymmetric metrics, generalized connections, nonholonomic and/or noncommutative structures became "equal in rights". An interference between gravitational and matter field equations and Ricci flow evolution equations results naturally in a new geometry and physics with a number of issues in classical and quantum gravity to be elucidated.

This paper and works [10, 11] must be considered as the first steps toward the implementations of a more general programme to understand the full dynamics and geometry of gravitational fields with symmetric and nonsymmetric metrics and generalized connections. Even we start with (standard) models of gravity with symmetric metrics, the Ricci flow theory "drive" us to nonholonomic configurations and nonsymmetric metrics. The next step is to elaborate the geometry of nonholonomic spaces enabled with (non)symmetric metric compatible connections [33].

Finally, it is worth noting that the present work can be extended to models with noncommutative and/or spinor variables and applied in modern astrophysics and cosmology for a study of scenaria with locally anisotropic/nonhomogeneous interactions. This information will be helpful in distinguishing gravity theories and fundamental spacetime and field interaction properties. Such subjects consist certain directions of our further investigations.

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